ISRAEL JOURNAL OF MATHEMATICS, Vol. 50, No. 4, 1985

NON-FREE TORSION-FREE PROFINITE GROUPS WITH OPEN FREE SUBGROUPS

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ABSTRACT
For every integer $e \geqq 3$ there exists a non-free torsion-free profinite group containing $\hat{F}_e$ as an open subgroup.

The aim of this note is to disprove the following

CONJECTURE (Jarden [3], Conjecture 4.4). Let $\mathscr{C}$ be a full family of finite groups and let $e \geqq 2$ be an integer. If a torsion-free pro-$\mathscr{C}$-group G contains an open subgroup F which is isomorphic to $\hat{F}_e(\mathscr{C})$ then G is a free pro-$\mathscr{C}$-group.

The Conjecture generalizes a result of Serre ([5], Corollaire 2): *Every torsion-free pro-p-group containing an open free subgroup is free.* Serre also asked whether the discrete analogue of this is true; a positive answer has been supplied by Stalling [6] and Swan [7]. Jarden has posed the profinite analogue as a conjecture in [2], section 13. Partial results with respect to the rank e of the free subgroup have been obtained since. The conjecture does not hold if e is infinite (Mel'nikov [4]); Jarden has proved the conjecture for $e = 2$ ([3], Theorem 1.2) and together with Brandis constructed a counterexample for $e = 1$ ([3], section 5), thereby stating the conjecture in the above-mentioned form. We claim:

PROPOSITION. *Let $e \geqq 3$. There exists a torsion-free non-free profinite group G with an open free subgroup H isomorphic to $\hat{F}_e$.*

PROOF. Choose primes p, q such that $p \mid e-1$, $p \mid q-1$ and let $n = (e-1)/p$. By [2], Example 5.1 there exists an exact sequence of profinite groups

†Supported by Rothschild Fellowship.
Received June 26, 1984 and in revised form September 2, 1984

$$1 \longrightarrow \hat{\mathbf{Z}} \longrightarrow \Gamma \xrightarrow{\varphi} \mathbf{Z}/p\mathbf{Z} \longrightarrow 1,$$

where Γ is torsion-free. Moreover, Γ possesses the following finite quotient

$$\bar{\Gamma} = \langle \bar{z}, \bar{\pi} \mid \bar{z}^q = \bar{\pi}^p = 1, \bar{z}^{\bar{\pi}} = \bar{z}^a \rangle,$$

with $a \in \mathbf{Z}$ such that $a^p \equiv 1 \pmod q$, $a \not\equiv 1 \pmod q$. (Indeed, $\Gamma = \langle z, \pi \rangle$, where $z = (z_l)_l$ generates $\Pi_{l \neq p} \mathbf{Z}_l$, π generates $\mathbf{Z}_p$, and $z_l^\pi = z_l^{\alpha_l}$ with $\alpha_l \in \mathbf{Z}_l$ chosen such that $\alpha_l^p = 1$ for every $l \neq p$, and $\alpha_q \neq 1$. Note that this implies that $\alpha_q \not\equiv 1 \pmod q$. Let $a \in \mathbf{Z}$ such that $a \equiv \alpha_q \pmod q$ and define an epimorphism $\Gamma \to \bar{\Gamma}$ by $z \mapsto \bar{z}$, $\pi \mapsto \bar{\pi}$.)

Now let G be the free profinite product $\hat{F}_n * \Gamma$, and define $\hat{\varphi} : G \to \mathbf{Z}/p\mathbf{Z}$ by $\mathrm{res}_{\hat{F}_n} \hat{\varphi} = 1$, $\mathrm{res}_\Gamma \hat{\varphi} = \varphi$. Let $H = \operatorname{Ker} \hat{\varphi}$. Then $G = H\Gamma$ and $G = \dot{\bigcup}_{i=1}^p x_i H = \dot{\bigcup}_{i=1}^p x_i H \hat{F}_n = \dot{\bigcup}_{i=1}^p H x_i \hat{F}_n$, where $x_1, \dots, x_p$ represent G/H. Moreover, $H \cap \Gamma = \operatorname{Ker} \varphi = \hat{\mathbf{Z}}$ and $H \cap \hat{F}_n^{x_i} \cong H \cap \hat{F}_n = \hat{F}_n$ for every i. Thus by the (Subgroup) Theorem of [1]

$$H \cong \hat{\mathbf{Z}} * \left(\coprod_{i=1}^{p} {}^{*} \hat{F}_n \right) * \hat{F}_m, \qquad \text{where } m = (p-1) + (p-p) - p + 1 = 0,$$

i.e., $H \cong \hat{F}_{pn+1} = \hat{F}_e$.

If $g \in G$ is of finite order then its image under the canonical projection $G \to \Gamma$ is 1, since Γ is torsion-free. In particular $\hat{\varphi}(g) = 1$, i.e., $g \in H$, whence $g = 1$, since $H \cong \hat{F}_e$ is torsion-free. If G were free, then

$$\mathrm{rk}(G) = \frac{\mathrm{rk}(H) - 1}{(G:H)} + 1 = \frac{pn}{p} + 1 = n + 1,$$

by the Corollary of [1]. However, we now construct a finite quotient $\bar{G}$ of G of rank $> n+1$:

Lemma. *Let $B = \mathbf{Z}/q\mathbf{Z} \times \cdots \times \mathbf{Z}/q\mathbf{Z}$ ($n+1$ times), and let $\bar{G} = \mathbf{Z}/p\mathbf{Z} \ltimes B$, where $\mathbf{Z}/p\mathbf{Z} = \langle c \rangle$ acts on B by*

$$b^c = b^a, \qquad \text{for all } b \in B.$$

Then:

(i) *If $1 \neq b \in B$ then $\langle b, c \rangle \cong \bar{\Gamma}$.*
(ii) *If $B' \leqq B$ then B' is normal in $\bar{G}$.*
(iii) *If $g \in \bar{G} \setminus B$ then* $\operatorname{ord} g = p$.
(iv) *$\bar{G}$ is a quotient of G.*
(v) $\mathrm{rk}(\bar{G}) > n+1$.

PROOF. (i), (ii) — clear.

(iii) As B is the kernel of the projection $\psi: \bar{G} \to \mathbf{Z}/p\mathbf{Z}$, we have $\psi(g) \neq 1$, hence $p = \operatorname{ord} \psi(g) \mid \operatorname{ord} g$. On the other hand $g = bc^i$, for some $b \in B$, $i \in \mathbf{Z}$. Thus $g \in \langle b, c \rangle \cong \bar{\Gamma}$, by (i), whence $\operatorname{ord} g \mid \operatorname{ord} \bar{\Gamma} = pq$. But $\bar{\Gamma}$ is of order pq and not cyclic, hence $\operatorname{ord} g \neq pq$. Therefore $\operatorname{ord} g = p$.

(iv) Choose generators $b_0, b_1, \ldots, b_n$ for B. Then $\langle b_1, \ldots, b_n \rangle$ is a quotient of $\hat{F}_n$ and $\langle b_0, c \rangle$ is a quotient of Γ, by (i), hence $\bar{G} = \langle c, b_0, b_1, \ldots, b_n \rangle$ is a quotient of G.

(v) Assume that $g_0, g_1, \ldots, g_n \in \bar{G}$ generate $\bar{G}$. W.l.o.g. $g_0 \notin B$. We may also assume that $g_1, \ldots, g_n \in B$, otherwise premultiply them by suitable powers of g_0. By (ii),

$$\bar{G} = \langle g_0 \rangle \langle g_1, \ldots, g_n \rangle = \langle g_0 \rangle (\langle g_1 \rangle \cdots \langle g_n \rangle),$$

hence $|\bar{G}| \leqq pq^n$, by (iii). A contradiction, since $|\bar{G}| = p\,|B| = pq^{n+1}$.

Note added in proof. The result of this note has been independently obtained by O. V. Mel'nikov.

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